# Fun/Interesting Problem Catalog

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# 1 Puzzles

## 1.1 Links to really great puzzle lists:

https://github.com/sharkdp/great-puzzles

## 1.2 Measurement, Weighing, and Physical Puzzles

- You have nine stones that look identical, but one of which contains a precious jewel inside of it. Eight of the stones weigh the same, and the stone with the jewel is slightly heavier than the others. You have a balance scale that you are only allowed to use twice. Is it possible to determine which is the slightly heavier stone containing the precious jewel from only two weighings? Explain why or why not.

- Once upon a time, a damsel was captured by an evil knight and imprisoned on a square island surrounded by a square moat. The moat was 20 feet across, and no drawbridge existed. A good knight found two sturdy beams of wood, each only 19 feet long and 8 inches wide. He had no nails, screws, saws, glue, or any other method of joining the two beams to extend their length. How did the maiden advise the knight to accomplish the rescue?

- At a very strange lemonade stand, you can buy lemonade only in precisely 8-oz servings, using only an unmarked 6-oz glass and an unmarked 10-oz glass. Is it possible to produce precisely 8 ounces from the large barrel using only these two glasses? If so, explain how; if not, explain why not.

- A creamer and a teacup each have exactly three ounces of cream and tea, respectively. One ounce of cream is poured into the tea and stirred until well mixed. Then one ounce of the mixture is put back into the creamer. Is the tea more diluted than the cream, is the cream more diluted than the tea, or are they equally diluted?

## 1.3 Logic and Deduction

- Three college students (majoring in math, English, and history) each have a colored dot (red or white) placed on their forehead. All three raise their hands when blindfolds are removed (indicating each sees at least one red dot). After some time, the math major hits the buzzer, knowing her dot color. Explain how she knew the color of her dot. Why did the other students not know?

- Congresswoman Smith says three of the five committee members always tell the truth. The members make the following statements:

  - Schlock: "It was either Wind or Pocket."
  - Wind: "Neither Slie nor I leaked the scandal."
  - Pocket: "Both of you (Schlock and Wind) are lying."
  - Greede: "One of them is lying and the other is telling the truth."

– Slie: "No, Greede, that is not true."

Assuming Smith's declaration is true, determine who leaked the beans.

- McGregor visits an island of knights (always truth) and knaves (always lie). He asks a shy old man, "Are both of you [you and your wife] knaves?" The man answers, "At least one of us is." What can McGregor infer about each member of the couple?
- McGregor knocks on a door; the husband says angrily, "We are both knaves!" and slams the door. What type is the husband and what type is the wife?
- A shy man says only: "If I am a knight, then so is my wife." What type is the husband and what type is the wife?

## 1.4 Games and Strategy

- Three missionaries and three cannibals need to cross a river using a boat that holds at most two people. At no time may cannibals outnumber missionaries on either side of the river (or in the boat). Is it possible for all six to cross? If so, explain how.
- Five rational, bloodthirsty pirates (A through E, in order of seniority) find 100 gold coins. The most senior proposes a distribution; all vote; majority wins, else the proposer is thrown overboard. Pirates prioritize: (1) survival, (2) gold, (3) throwing others overboard. How many coins does each pirate receive?
- Indiana Jones, his girlfriend, his father, and his father's sidekick must cross a rope bridge using one flashlight. The bridge holds at most two people; two must walk together at the slower person's speed. Their crossing times: Indiana 5 min, girlfriend 10 min, father 20 min, sidekick 25 min. Can they all cross in 60 minutes?

## 1.5 Miscellaneous Puzzles

- You are locked in a 50-by-50-by-50-foot room which sits on 100-foot stilts. There is an open window at the corner of the room, near the floor, with a strong hook cemented into the floor by the window. (The stilts are not accessible from the window.) There are two 50-foot lengths of rope, each cemented into the ceiling, about 1 foot apart, near the center of the ceiling. You are a strong, agile rope climber, good at tying knots, and you have a sharp knife, but no other tools. The rope is strong enough to hold your weight, but not if it is cut lengthwise. You die if you fall more than 10 feet. How do you get out alive?
- A group of jealous professors is locked up in a room with nothing but pencils and two tiny scraps of paper per person. The professors want to determine their average (mean) salary so that each one can compare their personal situation to their peers. However, they are secretive and do not want to reveal any personal salary information to anyone else. Can they determine the average salary in such a way that no professor can discover any fact about the salary of anyone but themselves before the average is revealed?

- https://en.wikipedia.org/wiki/Five-room_puzzle

- How many children? "I hear some youngsters playing in the garden," said Jones, an actuarial student. "Are they all yours?"

  "Heavens, no," exclaimed Professor Smith, the eminent actuary ... "My children are playing with friends from three other families in the neighbourhood, although our family happens to be the largest. The Browns have a smaller number of children, the Greens have a still smaller number, and the Blacks the smallest of all."

  "How many children are there altogether?" asked Jones.

  "Let me put it this way," said Smith. "There are fewer than 18 children, and the product of the numbers in the four families happens to be my house number which you saw when you arrived."

  Jones took a notebook and pencil from his pocket and started scribbling. A moment later he looked up and said, "I need more information. Is there more than one child in the Black family?"

  As soon as Smith replied, Jones smiled and correctly stated the number of children in each family. Knowing the house number and whether the Blacks had more than one child, Jones found the problem trivial.

  It is a remarkable fact, however, that the number of children in each family can be determined solely on the basis of the information given above. So ... can you work out how many children each family has?

# 2 Math Problems

## 2.1 Proofs and Number Theory

- A monk climbs a mountain. He starts at 8AM and reaches the summit at noon. He spends the night on the summit. The next morning, he leaves the summit at 8AM and descends by the same route, reaching the bottom at noon. Prove that there is a time between 8AM and noon at which the monk was at exactly the same spot on the mountain on both days. (The monk may travel at any speed, including stopping, going backward, etc.)

- How many subsets can be formed from a set of $n$ objects? Show that your answer is correct. (Include the empty set and the full set.)

- Show that for any three consecutive odd integers, one of them is divisible by 3.

- Find, with proof, all prime numbers $p$ and $q$ such that both $p + q$ and $p - q$ are also prime.

- If $n$ is a positive integer such that $2n + 1$ is a perfect square, show that $n + 1$ is the sum of two consecutive perfect squares.

- Prove that $\sqrt{3}$ is irrational.
- Prove that $\sqrt{6}$ is irrational.
- Prove that $\log_{10}(2)$ is irrational.
- Prove that if $a$ is rational and $b$ is irrational, then $a + b$ is irrational.
- True or false, and why: If $a$ and $b$ are irrational, then $a^b$ is irrational.

## 2.2 Combinatorics and Counting

- Define $s(n)$ as the number of ways to write the positive integer $n$ as an ordered sum of at least one positive integer. For example, $s(4) = 8$. Conjecture a general formula.
- Into how many regions is the plane divided by $n$ lines in general position (no two lines parallel; no three lines meet at a point)?
- How many subsets of $\{1, 2, 3, \ldots, 30\}$ have the property that the sum of the elements is greater than 232?
- A permutation $(a_1, a_2, \ldots, a_n)$ of the integers $1, 2, \ldots, n$ is *fantastic* if $a_1 + a_2 + \cdots + a_k$ is divisible by $k$ for each $k$ from 1 to $n$. Show that no fantastic permutation exists for $n = 4$. What about $n = 5$?

## 2.3 Algebra, Sequences, and Functional Problems

- Define $f(x) = \dfrac{1}{1-x}$ and denote $n$ iterations by $f^n(x)$. Find $f^{1999}(2000)$.
- The Fibonacci sequence is defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$. By exploring examples, find a simple formula for $(F_{n+1})^2 + (F_n)^2$.
- Carry out the long division of $1/n$ for various values of $n$ and determine the maximum possible size of the repeating block in the decimal expansion of $1/n$. Explain why a larger block is impossible.
- The Quakertown Poker Club plays with blue chips worth \$5.00 and red chips worth \$8.00. What is the largest bet that cannot be made with these chips?
- Prove that $7^n - 1$ is divisible by 6 for all positive integers $n$.
- Prove that
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
for all positive integers $n$.

## 2.4 Geometry and Measurement

- A rectangular room measures 30 feet in length, 12 feet in height, and 12 feet in width. A fly rests at a point one foot down from the ceiling at the middle of one end. A smudge of food is located one foot up from the floor at the middle of the other end. The fly has just enough energy to walk 40 feet. Show that there is a path along which the fly can walk to reach the food.

- Find the length of the shortest path from the point $(3, 5)$ to the point $(8, 2)$ that touches the $x$-axis and also touches the $y$-axis.

- A bug sits on one corner of a unit cube and wishes to crawl to the diagonally opposite corner, staying on the surface. What is the length of the shortest path?

- A rectangular box has faces with areas 15, 18, and 30. Find its volume and dimensions.

- A rectangular pan of brownies has had a smaller rectangular piece cut from it at an arbitrary position and angle. Find a single straight cut that divides the remaining brownies into two equal halves. Prove that your method always works.

## 2.5 Probability and Games (Mathematical)

- Walking down an upward-moving escalator, you count 150 steps. Your friend walks up and counts 75 steps. Your walking speed (in steps per unit time) is three times your friend's speed. How many steps are visible on the escalator at any given time?

- Assuming a two-person race, what is the smallest percentage of votes cast that a U.S. Presidential candidate could receive and still win? What about a three-person race? (State your assumptions.)

- Eve and Oddie flip 2005 coins. Eve wins if the number of heads is even; Oddie wins if odd. What is the probability that Eve wins?

## 2.6 Calculus and Analysis

- Determine
$$\int_1^5 \sin^{1001}(x-3)\,dx.$$

- Evaluate
$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx.$$

## 2.7 Extreme Principle

- Fifteen sheets of paper of various sizes and shapes lie on a desktop covering it completely. The sheets may overlap and may even hang over the edge. Show that five of the sheets may be removed so that the remaining ten sheets cover at least 2/3 of the desktop.

- Find all sets of five positive integers whose sum equals their product.
- At a party, no boy dances with every girl, but each girl dances with at least one boy. Show that there are two couples $bg$ and $b'g'$ that dance, whereas $b$ does not dance with $g'$ nor does $g$ dance with $b'$.
- Prove that dropping the lowest grade from a set of $n$ homework grades cannot decrease the average; that is, the new average is greater than or equal to the original one.
- Imagine an infinite chessboard that contains a positive integer in each square. If the value in each square is equal to the average of its four neighbors to the north, south, west, and east, prove that the values in all the squares are equal.
- Suppose you are given a finite set of coins in the plane, all with different diameters. Show that one of the coins is tangent to at most five of the others.
- On a large, flat field, $n$ people ($n > 1$) are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When $n$ is odd, show that there is at least one person left dry. Is this always true when $n$ is even?
- Place the integers $1, 2, 3, \ldots, n^2$ (without duplication) in any order onto an $n \times n$ chessboard, with one integer per square. Show that there exist two adjacent entries whose difference is at least $n + 1$. (Adjacent means horizontally or vertically or diagonally adjacent.)
- Let $p(x)$ be a real polynomial such that for all $x$, $p(x) + p'(x) \geq 0$. Show that for all $x$, $p(x) \geq 0$. What if $p(x)$ is instead any smooth function?
- Two drivers share a round trip. On the way there, the first driver drives the first 40 miles and the second drives the rest. On the way back (same route), the first driver drives the first leg and the second drives the last 50 miles. Who drove more, and by how many miles?
- Describe all infinite sequences $\{x_1, x_2, x_3, \ldots\}$ of real numbers such that for all $m, n$,

$$|x_n - x_m| \leq \frac{2mn}{m^2 + n^2}.$$

## 2.8 Pigeonhole Principle

- You have 10 pairs of socks, 5 black and 5 blue, all mixed up in a drawer. How many socks must you pull out (in the dark) to guarantee: (a) a matched pair of one color? (b) two matched pairs? (c) a matched pair of black socks?
- You want to buy a new car, and you know the model you want. The model comes in four different colors, and you can also choose to pay extra for heated seats (or not), a faster motor (or not), and a sunroof (or not). So far 100,000 cars of this model have been sold. What is the fewest number of these cars that must have the identical color and the same options?

- Prove that from a set of ten distinct two-digit numbers, it is possible to select two disjoint subsets whose members have the same sum.
- Prove that some multiple of $\sqrt{2}$ lies within 1/1,000,000 of an integer.
- Prove that in any group of six people there are either three mutual friends or three mutual strangers.
- Show that in any finite gathering of people, there are at least two people who know the same number of people at the gathering (assume that "knowing" is a mutual relationship).
- Seven points are placed inside a regular hexagon with side length 1. Show that at least two points are at most one unit apart.
- Choose any $(n+1)$-element subset from $\{1, 2, \ldots, 2n\}$. Show that this subset must contain two integers that are relatively prime.
- People are seated around a circular table at a restaurant. The food is placed on a circular platform in the center of the table, and this circular platform can rotate. Each person ordered a different entree, and it turns out that no one has the correct entree in front of him. Show that it is possible to rotate the platform so that at least two people will have the correct entree.
- Show that for any positive integer $n$, there exists a positive multiple of $n$ that contains only the digits 7 and 0.
- Pick any six elements from the set $\{1, 2, 3, \ldots, 10\}$. Show that you can always find two of these six elements that add up to 11.
- Five integers are picked at random from the set $\{1, 2, 3, \ldots, 2005\}$. Show that regardless of which five integers were chosen, it is always possible to select three of those five integers whose sum is divisible by three.

## 2.9 Parity

- Due to a labor dispute at McDonald's, employees have decided to sell Chicken McNuggets only in packages of 6 and 20. Is it possible to purchase exactly 2001 Chicken McNuggets?
- There are four jars in a room labeled A, B, C, and D. Jar A contains 9 marbles, Jar B contains 10 marbles, Jar C contains 13 marbles, and Jar D contains 16 marbles. Aaron plays the following game. He selects any three of the jars, draws one marble from each, and puts all three marbles into the fourth jar. He keeps repeating this. Can Aaron play in such a way so that eventually there will be exactly 12 marbles in each jar?
- Consider an 8-by-8 chessboard. Two diagonally opposite corner squares are removed, leaving 62 squares. Can the modified board be tiled with exactly 31 dominoes?

- Suppose you are given nine lattice points in three-dimensional Euclidean space. Show that the midpoint of at least one of the line segments joining two of these points is also a lattice point.

- Three frogs are placed on three vertices of a square. Every minute, one frog leaps over another frog, in such a way that the "leapee" is at the midpoint of the line segment whose endpoints are the starting and ending position of the "leaper." Will a frog ever occupy the vertex of the square that was originally unoccupied?

- Two people take turns cutting a rectangular chocolate bar that is $6 \times 8$ squares in size. Cuts must be straight and along a division between squares. The last player to make a cut wins. Is there a winning strategy for the first or second player? What about the general $m \times n$ case?

- Show that if every room in a house has an even number of doors, then the number of outside entrance doors must be even as well.

- There are 7 coins on a table, all tails up. You can flip over exactly two at a time. Can you make all coins heads up?

- In a large urn there are 1999 orange balls and 2000 yellow balls. Next to the urn is a large pile of yellow balls. Repeatedly choose two balls from the urn and apply the stated replacement rules. What is the color of the last ball in the urn?

- Place a knight on each square of a 7-by-7 chessboard. Is it possible for each knight to simultaneously make a legal move?

- An evil wizard has imprisoned 64 math geeks. The wizard will stand them in a line and place a black or white hat on each person's head. Each person can see the hats of everyone in front of them but not their own or those behind. Starting from the back of the line, each person must say only "black" or "white"; a correct guess earns a cookie, an incorrect guess is fatal. The prisoners may agree on a strategy beforehand but may not communicate once in line (though they can hear prior guesses and outcomes). Describe a strategy that guarantees at most one person is killed.

- Lockers numbered 1 to 100 stand in a row at the school gym. When the first student arrives, she opens all the lockers. The second student then goes through and recloses all the even-numbered lockers; the third student changes the state of every locker whose number is a multiple of 3. This continues until 100 students have passed through. Which lockers are now open?

## 2.10 Algebraic Manipulation

- Find all positive integer solutions $(x, y)$ to $x^2 - y^2 = 20$.

- Find $3x^3y^2$ if $x, y$ are integers such that $y^2 + 3x^2y^2 = 30x^2 + 517$.

- Show that $n^4 - 20n^2 + 4$ is composite for every integer $n$.

- Show that for no positive integer $n$ can both $n+3$ and $n^2+3n+3$ be perfect cubes.
- Find all integer solutions $(x, y)$ to $xy + 5x + 3y = 200$.
- Suppose $a, b, c, d$ are real numbers satisfying

$$a^2 + b^2 = 1, \quad c^2 + d^2 = 1, \quad ac + bd = 0.$$

  Show that

$$a^2 + c^2 = 1, \quad b^2 + d^2 = 1, \quad ab + cd = 0.$$

- Find $x^2 + y^2$ if $x, y$ are natural numbers, $xy + x + y = 71$, and $x^2y + xy^2 = 880$.
- Find all real values $x$ that satisfy

$$(16x^2 - 9)^3 + (9x^2 - 16)^3 = (25x^2 - 25)^3.$$

- Determine all real numbers $x, y, z, w$ that solve the system

$$x + y + z = w, \qquad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{w}.$$

## 2.11 Sums and Products

- Find a formula for the product of the terms of a geometric sequence $a, ar, ar^2, \ldots, ar^n$.
- Find the sum

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}+\sqrt{n+1}}.$$

- Prove the identity

$$\sum_{k=1}^{n} (k^2+1)k! = n \cdot (n+1)!.$$

- Find the sum

$$\sum_{k=2}^{n} \frac{1}{\log_k u}.$$

- Find a closed formula for the sum

$$\sum_{k=1}^{n} \frac{k}{(k+1)!}.$$

- Evaluate the product

$$\prod_{k=0}^{n} \cos(2^k \theta).$$

- Prove that
$$\sum_{n=1}^{9999} \frac{1}{(\sqrt{n}+\sqrt{n+1})(\sqrt[4]{n}+\sqrt[4]{n+1})} = 9.$$

- Find the sum
$$1 + (1+2) + (1+2+3) + \cdots + (1+2+3+\cdots+n).$$

- A 2-inch elastic band is fastened to the wall at one end, with a bug at the other end. Every minute the band is uniformly stretched by 1 inch, and then the bug walks 1 inch toward the fastened end. Will the bug ever reach the wall?

## 2.12 Polynomials

- Prove that a non-zero polynomial of degree $n$ can have at most $n$ distinct zeros.
- Find a polynomial with integer coefficients whose zeros include $\sqrt{2}+\sqrt{5}$.
- Let $P(x)$ be a polynomial with integer coefficients satisfying $P(0) = P(1) = 1999$. Show that $P(x)$ has no integer zeros.
- Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.
- Prove that if two polynomials $P(x)$ and $Q(x)$, both of degree $d$ or less, agree at $d+1$ distinct values of $x$, then $P(x) = Q(x)$.
- $P(x)$ is a polynomial with remainders $a$, $b$, and $c$ when divided by $x-a$, $x-b$, and $x-c$ respectively. What is the remainder when $P(x)$ is divided by $(x-a)(x-b)(x-c)$?
- Verify that $\sqrt[3]{20+14\sqrt{2}} + \sqrt[3]{20-14\sqrt{2}} = 4$.
- Prove that there does not exist a polynomial $f(x)$ with integer coefficients for which $f(2020) = 0$ and $f(2022) = 2021$.
- Let $P(x)$ be a monic polynomial with integer coefficients such that $P(a) = P(b) = P(c) = P(d) = 5$ for four distinct integers $a, b, c, d$. Prove there is no integer $k$ with $P(k) = 8$.

## 2.13 Inequalities

- True or false? Give counterexamples if false. Assume all variables are real numbers.
  - If $x < y$, then $xz < yz$.
  - If $|x| < 3$, then $\dfrac{|x+1|}{|x-2|} < 3$.
  - If $|x-y| < M$, then $|x| < |y| + M$.

- If $x, y$ are positive numbers, show that

$$\sqrt{\frac{x^2 + \left(\frac{x+y}{2}\right)^2}{2}} \leq \frac{x + \sqrt{\frac{x^2+y^2}{2}}}{2}$$

  and find all cases of equality.

- Prove that $n! < \left(\frac{n+1}{2}\right)^n$ for $n = 2, 3, 4, \ldots$.

- Prove that for all integers $n \geq 1$,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|.$$

- Prove the reverse triangle inequality: $|x - y| \geq \big||x| - |y|\big|$.

- Show that if $c > 0$ and $|x - 2| < c$, then $|x^2 - 4| < c^2 + 4c$.

- Let $a_1, a_2, \ldots, a_n$ be any sequence of numbers and $b_1, b_2, \ldots, b_n$ be any permutation of the first sequence. Prove that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} \geq n.$$

- Prove that

$$1^1 \cdot 2^2 \cdot 3^3 \cdots n^n < \left(\frac{2n+1}{3}\right)^{\frac{n(n+1)}{2}}$$

  for all $n = 2, 3, 4, \ldots$.

- Prove that

$$0 \leq xy + yz + zx - 2xyz \leq \frac{7}{27},$$

  where $x, y, z$ are non-negative real numbers with $x + y + z = 1$.

## 2.14 Counting

- Eight boys and nine girls sit in a row of 17 seats.
  - How many different seating arrangements are there?
  - How many different seating arrangements are there if all the boys sit together and all the girls sit together?
  - How many seating arrangements are there if no one sits next to anyone of the same gender?

- Between 1 and 1,000,000 inclusive, which are more numerous: integers whose base-ten representation contains a "1", or those that do not? How many of each type are there?

- A typical small packet of Skittles includes 24 Skittles in five different flavors, not necessarily equally distributed. How many different packs of Skittles are possible?
- Each side of a square is subdivided into 101 segments. Find the number of triangles whose vertices lie at these 404 subdivision points.
- Among natural numbers from 1 to 1,000,000, are there more that can be represented as a sum of a perfect square and a positive perfect cube, or more that cannot?
- Two of the squares of a 7-by-7 checkerboard are painted yellow, and the rest are painted green. Two color schemes are equivalent if one can be obtained from the other by a rotation of the board. How many inequivalent color schemes are possible?
- In a class there are 10 boys and 10 girls.
  - The teacher pairs each boy with a girl. In how many ways can this be done?
  - The teacher pairs the students into 10 partnerships with no gender restriction. In how many ways can this be done?
  - Same as (b), but the teacher also assigns each partnership one of 10 distinct report topics. In how many ways can this be done?
- A subset $\{a_1, a_2, \ldots, a_k\}$ of $\{1, 2, \ldots, n\}$ is said to be *separated* if $a_{i+1} - a_i \geq 2$ for $i = 1, 2, \ldots, k-1$. Show that the number of $k$-element separated subsets of $\{1, 2, \ldots, n\}$ is $\binom{n-k+1}{k}$.
- A set of eleven bank robbers decided to store their loot in a safe with multiple locks. The safe should open if and only if a majority of the robbers are present. How many locks are required, and how many keys must each robber carry?

## 2.15 Primes and Divisibility

- Pick two 2-digit numbers. Find their greatest common divisor using the Euclidean Algorithm. Explain why the algorithm works.
- Prove that the fraction $\dfrac{21n+4}{14n+3}$ is in lowest terms for every positive integer $n$.
- Show that consecutive odd integers are relatively prime.
- Find the smallest positive integer $n$ such that $n^2$ is a perfect square, $n^3$ is a perfect cube, and $n^5$ is a perfect fifth power.
- True or false, and why: The product of two consecutive positive integers cannot be a perfect square.
- Prove that the fraction $\dfrac{n^3+2n}{n^4+3n^2+1}$ is in lowest terms for every positive integer $n$.

- Show that 1000! ends in exactly 249 zeros.
- Prove that among any 12 consecutive positive integers there is at least one which is smaller than the sum of its proper divisors.
- True or false: The product of three consecutive positive integers cannot be a perfect square.
- Prove that any integer greater than or equal to 7 can be written as a sum of two relatively prime integers, both greater than 1.

## 2.16 Congruence

- What are the last two digits of $77^{77}$?
- Prove that $3333^{4444} + 4444^{3333}$ is divisible by 7.
- Do there exist four consecutive positive integers each of which can be written as the sum of two nonzero perfect squares?
- Define a sequence by $a_1 = 2$ and $a_{n+1} = a_n + 3$ for $n \geq 1$. Do there exist indices $p, q, r$ such that $a_p \cdot a_q = a_r$?
- Show that if $a^2 + b^2 = c^2$, then $3 \mid ab$.
- Let $N$ be a nine-digit number using each of the digits 1–9 exactly once such that, for each $k$ from 1 to 9, the first $k$ digits of $N$ form a number divisible by $k$. Find $N$.
- The number $2^{29}$ has 9 distinct digits. Without using a calculator, determine which digit is missing.
- Prove that any subset of 55 numbers chosen from $\{1, 2, 3, \ldots, 100\}$ must contain two numbers differing by 9.
- The number $d_1 d_2 \cdots d_9$ has nine decimal digits. The number $e_1 e_2 \cdots e_9$ is such that each of the nine 9-digit numbers formed by replacing just one digit $d_i$ by $e_i$ is divisible by 7. The number $f_1 f_2 \cdots f_9$ is related to $e_1 e_2 \cdots e_9$ in the same way. Show that $d_i - f_i$ is divisible by 7 for each $i$.